Comment to audience: Forget what's a monoid!

Monoids

Monoid Actions

straightening / unstraightening explicit

↳ categorical monoid actions

Categories with one object

nerve as higher

$\Delta^{op} \to$ sets categorical + conditions construction

straightening / unstraight.

monoidal categories.

The goal of this talk is to explain a categorical perspective on monoids.
First let's do the standard definition.

Definition: A monoid $(M,\cdot)$ consists of

Data:
- a set M
- a map $\cdot : M \times M \to M$, $(m_1,m_2) \mapsto \underset{\substack{!! \\ m_1 \cdot m_2}}{m_1 \cdot m_2}$

such that the following hold

Axioms:
- $\forall m_1,m_2,m_3 \in M \ (m_1 \cdot m_2) \cdot m_3 = m_1 \cdot (m_2 \cdot m_3)$
- $\exists 1 \in M \ \forall m \in M \quad m \cdot 1 = m = 1 \cdot m$

Example: -1) $(\{1\},\cdot)$ forms a monoid, called trivial monoid.
0) $(\mathbb{Z},+)$ forms a monoid.
1) Every group $(G,\cdot)$ forms a monoid.
2) Let $M = \{1, s, \delta_1, \delta_2\}$ with the following multiplication:

$1 \cdot x = x = x \cdot 1$ $\quad \delta_1^2 = \delta_1 \quad \delta_1\delta_2 = \delta_1 \quad s\delta_1 = \delta_2 \quad s\delta_2 = \delta_1$
$\forall x \in M \quad \delta_2^2 = \delta_2 \quad \delta_2\delta_1 = \delta_2 \quad \delta_1 s = \delta_1 \quad \delta_2 s = \delta_2$
$s^2 = 1$

this defines a monoid, which is not a group.

3) Given a monoid $(M, \cdot)$, there is the ==opposite monoid== M^{op}, whose set is M and $m_1 \cdot_{op} m_2 := m_2 \cdot m_1$.

4) Let X be a set. The pair $(\mathrm{Maps}(X, X), \circ)$ forms a monoid.

composition of maps

5) Let $\mathcal{C}$ be a category. Let $c \in \mathcal{C}$ be an object. The pair $(\mathrm{End}_{\mathcal{C}}(c), \circ)$ forms a monoid called

composition in $\mathcal{C}$.

the ==endomorphism monoid== of c.

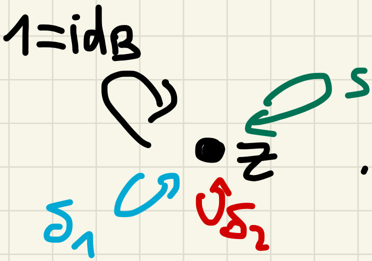
Remark: The next construction shows that every monoid appears in this way:

Definition: Let $(M, \cdot)$ be a monoid. Define the classifying category BM of M with

$ob(BM) = \{Z\}$ unique object

$Hom_{BM}(Z,Z) := M$, $\circ := \cdot$ composition is multiplication in M.

Example: 1) Let $M = \{1, s, \delta_1, \delta_2\}$ one might draw BM as



Some people call BM the delooping of M.

2) We have by definition

$$(BM)^{op} = BM^{op}$$

opposite category

Definition: Let $(M, \cdot)$ be a monoid.

Let X be a set. A left action of M on X is

Data: [A map $M \times X \longrightarrow X$, $(m, x) \longmapsto m \cdot x$

Axioms: [$\forall m_1, m_2 \in M \ \forall x \in X \ (m_1 \cdot m_2) \cdot x = m_1 \cdot (m_2 \cdot x)$

$\forall x \in X \ 1 \cdot x = x$.

Example | Definition:

Let $(M, \cdot)$ be a monoid.

0) M acts one a one-point set $\{0\}$ by setting $m \cdot 0 = 0$ for all $m \in M$.

We call this the trivial M-set.

1) M acts on M by setting
$\underset{\text{action}}{m \cdot m'} := \underset{\text{multiplication in } M}{m \cdot m'}$

we call this the regular M-set / the regular M-action.

2) Assume M acts on X. Let $Y \subseteq X$ be a subset s.t. $\forall m \in M \ \forall y \in Y \ m \cdot y \in Y$ (we call Y a M-subset of X). Then M acts on Y.

3) Let $M = \{1, s, \delta_1, \delta_2\}, \cdot$ be the monoid from before. View it as M-set via the regular action.
The subset $Y = \{\delta_1, \delta_2\} \subseteq M$ is an M-subset and hence an M-set.

4) Let X be a set. Then $(\mathrm{Maps}(X,X), \circ)$ acts on X via $f \cdot x := f(x)$
$f \in \mathrm{Maps}(X,X)$, $x \in X$.

Definition: Let $(M,\cdot)$, $(M',\cdot)$ be monoids.

A monoid morphism from M to M' is

Data: A map $f: M \longrightarrow M'$

Axioms: $\forall m_1, m_2 \in M: f(m_1 \cdot m_2) = f(m_1) \cdot f(m_2)$.
$f(1) = 1$.

A monoid isomorphism from M to M' is a bijective monoid morphism.

Example: Let $(M,\cdot)$ be a monoid.

0) The constant map $p: M \longrightarrow \{1\}$ is a monoid morphism.

1) The identity map $id_M: M \longrightarrow M$ is a monoid isomorphism $(M,\cdot) \longrightarrow (M,\cdot)$

2) The identity map $id_M: M \longrightarrow M^{op}$ is a monoid isomorphism $(M,\cdot) \longrightarrow (M,\cdot_{op})$ if and only if M is abelian $\forall m_1, m_2 \in M$ $m_1 \cdot m_2 = m_2 \cdot m_1$

3) Let $(G,\cdot)$ be a group. Then

$$(\,)^{-1}: G \longrightarrow G^{op}$$
$$g \longmapsto g^{-1}$$

is a monoid isomorphism

4) Let $M = \{1, s, \delta_1, \delta_2\}, \cdot$ from before. There is no monoid isomorphism $M \to M^{op}$

[<u>Indeed:</u> $\delta_1 \in M$ satisfies $\forall m \in M$ $\delta_1 m = \delta_1$

there is no element $x \in M$ s.t. $\forall m \in M$ $m \cdot x = x$].

An isomorphism $M \to M^{op}$ would map δ_1 to such x.

5) Let $\mathcal{C}, \mathcal{D}$ be categories, $F: \mathcal{C} \longrightarrow \mathcal{D}$ a functor.

Let $c \in \mathcal{C}$ be an object. Then

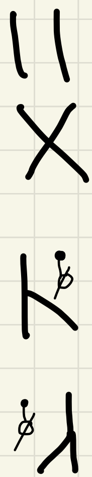
$$\begin{aligned} F: \mathrm{End}_{\mathcal{C}}(c) &\longrightarrow \mathrm{End}_{\mathcal{D}}(F(c)) \\ f &\longmapsto F(f) \end{aligned}$$

is a monoid morphism.

6) There is a monoid isomorphism

$$(M = \{1, s, \delta_1, \delta_2\}, \cdot) \longrightarrow (\mathrm{Maps}(\{1,2\},\{1,2\}), \circ)$$

$$\begin{aligned} 1 &\longmapsto \mathrm{id}_{\{1,2\}} \\ s &\longmapsto \begin{pmatrix} 1 \mapsto 2 \\ 2 \mapsto 1 \end{pmatrix} \\ \delta_1 &\longmapsto \begin{pmatrix} 1 \mapsto 1 \\ 2 \mapsto 1 \end{pmatrix} \\ \delta_2 &\longmapsto \begin{pmatrix} 1 \mapsto 2 \\ 2 \mapsto 2 \end{pmatrix} \end{aligned}$$



7) Let $(M, \cdot)$ be a monoid acting on a set X.

The action morphism
$$M \longrightarrow \mathrm{Maps}(X, X)$$
$$m \longmapsto (m \cdot \_ : x \mapsto m \cdot x)$$
is a monoid morphism.

The following lemma phrases this more precisely.

Lemma: Let $(M,\cdot)$ be a monoid, let X be a set. The $1:1$-correspondence (adjunction/currying)

$$\mathrm{Maps}(M \times X, X) \overset{1:1}{\longleftrightarrow} \mathrm{Maps}(M, \mathrm{Maps}(X,X))$$

$$f \longmapsto [m \mapsto (x \mapsto f(m,x))]$$

$$[(m,x) \mapsto g(m)(x)] \longmapsfrom g$$

restricts to a bijection

$$\left\{ \begin{array}{l} \text{Actions of } (M,\cdot) \\ \text{on } X \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{monoid morphisms} \\ f: M \to \mathrm{Maps}(X,X) \end{array} \right\}.$$

Under this bijection an action corresponds to it's action morphism.

Proof: Since $\overset{1:1}{\longleftrightarrow}$ is a bijection, it is enough to show that these two subsets are mapped under the bijection onto each other.

"$\Rightarrow$" Let $\cdot$ be an action of $(M, \cdot)$ on X.

Let $m_1, m_2 \in M$. Let $x \in X$

$$((m_1 \cdot -) \circ (m_2 \cdot -))(x) \overset{\text{Def}}{=} m_1 \cdot - ((m_2 \cdot -)(x))$$
$$= (m_1 \cdot -)(m_2 \cdot x)$$
$$= m_1 \cdot (m_2 \cdot x)$$
$$= (m_1 \cdot m_2) \cdot x$$
$$= ((m_1 \cdot m_2) \cdot -)(x)$$

Hence the action morphism $m \mapsto (m \cdot -)$ is a monoid morphism. (Also $(1 \cdot -)(x) = 1 \cdot x = x$
$= id_X(x)$)

"$\Leftarrow$" Let $f: M \to Maps(X, X)$ be a monoid morphism. We have to check that $m \cdot_f x := f(x)(m)$ defines an action.

Indeed let $m_1, m_2 \in M$, $x \in X$. Then

$$m_1 \cdot_f (m_2 \cdot_f x) \overset{\text{Def}}{=} m_1 \cdot_f (f(m_2)(x))$$

$$\overset{\text{Def}}{=} f(m_1)(f(m_2)(x))$$

$$\overset{\text{Def composition}}{=} (f(m_1) \circ f(m_2))(x)$$

$$\underset{\text{morphism}}{\overset{f \text{ is monoid}}{=}} (f(m_1 \cdot m_2))(x)$$

$$\overset{\text{Def}}{=} (m_1 \cdot m_2) \cdot_f x$$

Also $1 \cdot x \overset{\text{Def}}{=} f(1)(x) \underset{\substack{f \text{ is} \\ \text{monoid} \\ \text{morphism}}}{=} \mathrm{id}_X(x) = x.$

Hence it really defines an action. $\square$

Definition: Let $(M, \cdot)$ be a monoid and $(X, \cdot)$ be a M-set.
The universal category UX of X is is the category with

objects: $ob(UX) = X$

morphisms: $\mathrm{Hom}_{UX}(x_1, x_2) = \{ m \in M \mid m \cdot x_1 = x_2 \}$

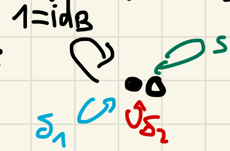
composition: Let $x_1 \xrightarrow{m_1} x_2$, $x_2 \xrightarrow{m_2} x_3$ be morphisms. Define $m_2 \circ m_1 := m_2 \cdot m_1 \in M$

Note: this is indeed an element in $\mathrm{Hom}_{UX}(x_1, x_3)$ since

$$(m_2 \cdot m_1) \cdot x_1 \overset{\text{Action ass.}}{=} m_2 \cdot (m_1 \cdot x_1) \overset{\text{Def}}{=} m_2 \cdot x_2 \overset{\text{Def}}{=} x_3.$$

$id_x = 1 \in M.$

Example: 1) Consider the trivial action of $M = \{1, s, \delta_1, \delta_2\}$ on $X = \{0\}$

Picture of $U\{0\}$ = 

2) Let $(M, \cdot)$ be a monoid.

Let $X = \{0\}$ be the trivial X-set.

Then we have an equivalence

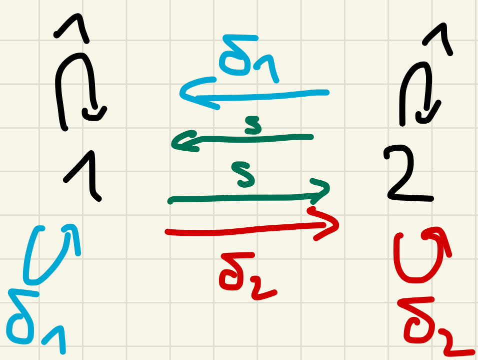
$U\{0\} \simeq BM$ given by

calling 0 z.

3) Consider the action of

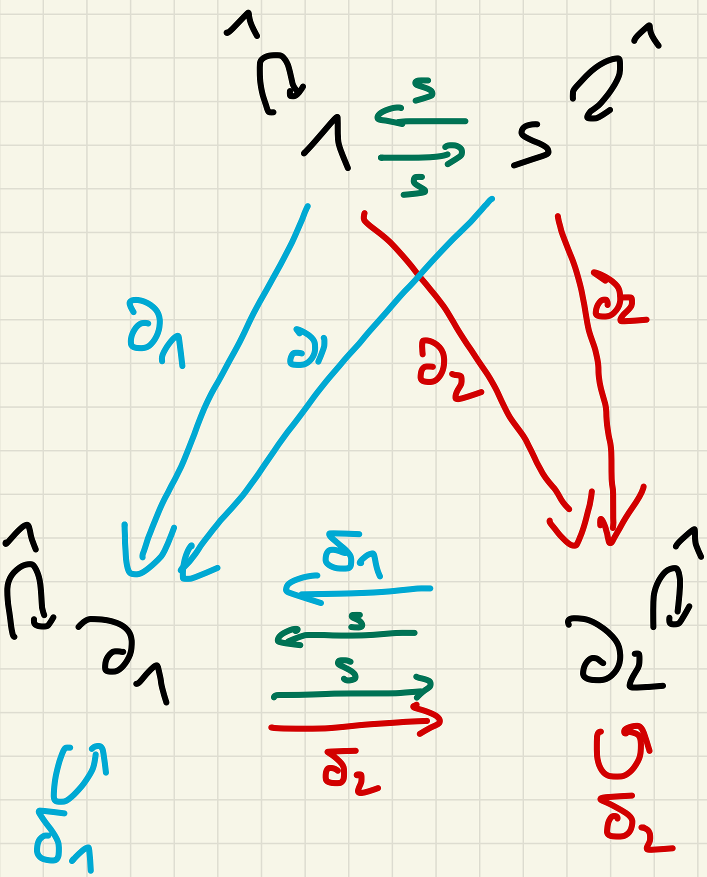
$M = \{1, s, \delta_1, \delta_2\}$ on $X = \{1, 2\}$

Then UX looks as follows:



4) Consider the regular action of $M = \{1, s, \partial_1, \partial_2\}$ on M.

Then UM looks as follows:



5) Let G be a group.

Let X be a G-set. Then $\mathcal{U}X$ is a groupoid. ← some call it the action groupoid of X

⌜Indeed $x_1 \xrightarrow{g} x_2$ has inverse $x_2 \xrightarrow{g^{-1}} x_1$

Moreover there is a bijection:

$$\underbrace{\{\text{objects in } \mathcal{U}X\}/\underset{\text{isomorphism}}{\cong}}_{=: \pi_0(\mathcal{U}X)} \overset{1:1}{\longleftrightarrow} G\text{-orbits of } X.$$

6) special case: Let G be a group.

Then BG and $\mathcal{U}G$ are groupoids

There is a unique iso-class in BG and $\mathcal{U}G$ given by $*$ which corresponds to the fact that the trivial/regular G-sets have 1-orbit.

Definition: Let $(M, \cdot)$ be monoids, X, Y be M-sets. A morphism $X \to Y$ of M-sets is

Data: A map $f: X \longrightarrow Y$

Axiom: $\forall m \in M \ \forall x \in X \quad f(m \cdot x) = m \cdot f(x)$.

We write M-Set for the category of M sets

Example: 1) Let $(M, \cdot)$ be any monoid, X an M-set and $\{0\}$ be the trivial M-set. The constant map

$$p : X \longrightarrow \{0\}$$ is a morphism of M-sets.

2) Let $Y \subseteq X$ be a M-subset. The inclusion

$$\iota : Y \longrightarrow X$$ is a morphism of M-sets.

Definition: Let $(M,\cdot)$ be a monoid.

X, Y M-sets, $X \xrightarrow{f} Y$ morphism of M-sets.

We define a functor

$$Uf: \quad UX \longrightarrow UY$$ given by

on objects: $(Uf)(x) := f(x)$

on morphisms: $(Uf)(x \xrightarrow{m} x') := (f(x) \xrightarrow{m} f(x'))$

⌜ **Note:** This morphism in the image exists since

$$m \cdot f(x) \overset{f \text{ morphism of } M\text{-sets}}{=} f(m \cdot x) \underset{\text{Ass.}}{=} f(x')$$ ⌟

Example / Definition:

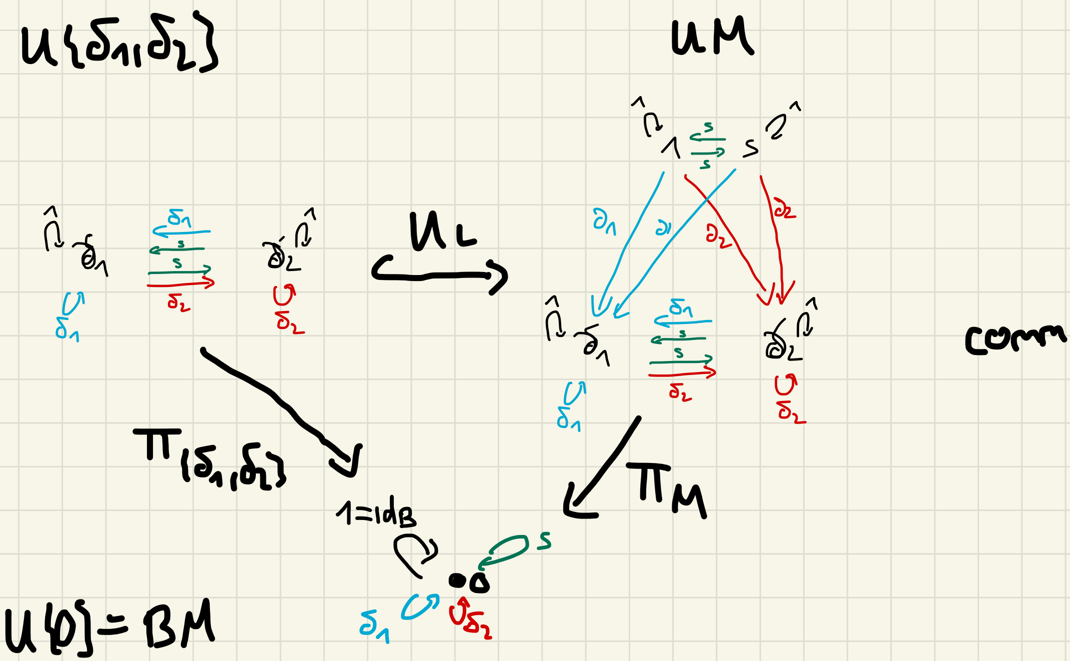
1) Let $(M, \cdot)$ be a monoid, X be a M-set.
The constant map $p : X \longrightarrow \{0\}$
induces a faithful functor

$$\pi_X := Up : UX \longrightarrow U\{0\} = BM$$

called **friendly projection**.

2) Let $M = \{1, s, \partial_1, \partial_2\}$

we saw three M-sets: $\{0\}$, $\{\delta_1, \delta_2\} \hookrightarrow M$.



Definition:

We define a functor

$$\begin{array}{rcl} U: M\text{-Set} & \longrightarrow & Cat_{/BM} \\ X & \longmapsto & \left(\begin{array}{c} UX \\ \downarrow \pi_X \\ BM \end{array}\right) \end{array}$$

1-slice category of small categories over BM.

$$\left(X \xrightarrow{f} Y\right) \longmapsto \left(\begin{tikzcd} UX \arrow[rr, "Uf"] \arrow[dr, "\pi_X"'] & {} \arrow[d, phantom, "\equiv"] & UY \arrow[dl, "\pi_Y"] \\ & BM & \end{tikzcd}\right)$$

We call U the unstraightening.

Lemma:

U is fully faithful.

Proof: Faithful:

Let $X \overset{f_1}{\underset{f_2}{\rightrightarrows}} Y$ be two morphism of M-sets

with $f_1 \neq f_2$, i.e. $\exists x \in X \; f_1(x) \neq f_2(x)$

But then $(Uf_1)(x) = f_1(x) \neq f_2(x) = (Uf_2)(x)$

Hence Uf_1 and Uf_2 are different functors.

$$Uf_1 \neq Uf_2.$$

<u>Full:</u>

Let X, Y be M-sets. Let $F : UX \to UY$ a functor over BM. In particular F consists of a map $X = ob(UX) \to ob(UY) = Y$, call this map f.

We claim f is a morphism of M-sets.

Let $m \in M$ $x \in X$. In UX there is a morphism $x \xrightarrow{m} m \cdot x$.

Applying F we get a morphism $F(m)$ in UY.

$$\begin{tikzcd}
UX & & & UY \\
x \arrow[r, "m"] & m\cdot x \arrow[r, squiggly, "F"] & f(x) \arrow[r, "F(m)"] & f(m\cdot x) \\
& z \arrow[r, "m"] & z & BM
\end{tikzcd}$$

π_X , π_Y

, for some $m' \in M$.

Since $z \xrightarrow{m} z = \pi_X(x \xrightarrow{m} m\cdot x)$

$$= (\pi_Y F)(x \xrightarrow{m} m\cdot x)$$

$$= \pi_Y(f(x) \xrightarrow{F(m)} f(m\cdot x))$$

$$= z \xrightarrow{F(m)} z$$

$\Rightarrow \quad m : f(x) \longrightarrow f(m\cdot x)$ in UY,

which means by definition

$$f(m\cdot x) = m\cdot f(x)$$

$\Rightarrow f : X \longrightarrow Y$ is morphism of M-sets, moreover it is clearly mapped to

$$F : UX \longrightarrow UY$$

$\Rightarrow U$ is full. $\blacksquare$

Remark: The following lemma describes the essential image of the functor U.

Lemma: Let $\begin{matrix} \mathcal{U} \\ \pi \downarrow \\ BM \end{matrix} \in Cat_{/BM}$. The following are equivalent:

i) $\exists X$ M-set and iso

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\exists \cong"] \arrow[dr, "\pi"'] & & \mathcal{U}X \arrow[dl, "\pi_X"] \\ & BM & \end{tikzcd}$$

for some M-set X

isomorphism of categories over BM

ii) $\forall m \in M$ $\forall u \in \mathcal{U}$ $\exists! u' \in \mathcal{C}$ $\exists!\ u \xrightarrow{f} u'$ such that $\pi(f) = m$

$$\begin{tikzcd} u \arrow[r, "\exists! f"] \arrow[d, "\pi", maps to] & \exists!\, u' \arrow[d, "\pi", maps to] \\ z \arrow[r, "\pi(f) = m"] & z \end{tikzcd}$$

$c \xrightarrow{1} c'$

We call such $\pi : \mathcal{U} \to BM$ a cocartesian fibration.

we call $u' := m \cdot u$ and write $u \xrightarrow{m} u'$ for f.

Proof: i) $\Rightarrow$ ii) Clearly $\begin{matrix} \mathcal{U}X \\ \pi_X \downarrow \\ BM \end{matrix}$ satisfies this property $m \in M$, $x \in X$ $x' = m \cdot x$ $x \xrightarrow{m} x'$

Moreover this property is preserved under isos of categories.

ii) $\Rightarrow$ i) Given such $\begin{array}{c} \mathcal{U} \\ \pi \downarrow \\ X \end{array}$

we define $X := \mathrm{ob}\,\mathcal{C}$

and an action of M on X by setting

$m \cdot c = c'$ $\leftarrow$ given by property ii).

Note that this indeed an action:

- We have $u \xrightarrow{\mathrm{id}_u} u$ which is mapped under f to $z \xrightarrow{1} z$ which shows by uniqueness that $1 \cdot c = c$.

$u \longrightarrow \exists!\, 1u$

- Let $m_1, m_2 \in M$ $u \in \mathcal{U}$ want to show $(m_1 \cdot m_2) \cdot cu \overset{!!}{=} m_1 \cdot (m_2 \cdot cu)$

$$cu \xrightarrow{m_2} m_2 \cdot cu \xrightarrow{m_1} m_1 \cdot (m_2 \cdot cu)$$

$$cu \xrightarrow{m_1 \cdot m_2} (m_1 \cdot m_2) \cdot cu$$

Now $\pi(m_1 \circ m_2) = \pi(m_1) \circ \pi(m_2)$

$= m_1 \cdot m_2$ in M.

Uniqueness

$\Rightarrow \quad (m_1 \cdot m_2) \cdot a = m_1 \cdot (m_2 \cdot a)$

In total we see that X is M-set.

By direct inspection the functor

$$\begin{aligned} \mathcal{U} &\longrightarrow \mathcal{U}X. \\ u &\longmapsto u \\ (u \xrightarrow{f} u') &\longmapsto (u \xrightarrow{\pi(f)} u') \end{aligned}$$

$\pi(f) \in M$

is an isomorphism of categories. $\blacksquare$.

Definition:

We write $\mathrm{CoCart}(BM) \subseteq \mathrm{Cat}_{/BM}$ for the full subcategory of $\mathrm{Cat}_{/BM}$ whose objects are cocartesian fibrations over BM.

Question: What are categories

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \mathbb{B}M \end{array}$$

over $\mathbb{B}M$ equivalent to (but not necessarily isomorphic to) a friendly projection

$$\pi_X \quad \begin{array}{c} UX \\ \downarrow \\ \mathbb{B}M \end{array} \quad ?$$

Remark: Let $(M, \cdot)$ be a monoid.
There is an isomorphism of categories

$$\text{M-Set} \xrightarrow{\cong} \underbrace{\mathrm{Fun}(BM, \mathrm{Sets})}_{\text{functor category}}$$

Proof: Let $(X, \cdot)$ be a M-set.
Define a functor $F_X : BM \to \mathrm{Set}$
by setting
$$F_X(z) = X$$
$$F_X(m) = m \cdot \_ : X \to X$$

(**Note:** F_X is indeed a functor, since the action morphism is a monoid morphism).

Given two M-sets X, Y and a morphism $f : X \to Y$ of M-sets we define a natural transformation

$\alpha_F : F_X \Longrightarrow F_Y$ by setting

$$(\alpha_F)_B : \underset{\displaystyle X}{\underset{\shortparallel}{F_X(Z)}} \longrightarrow \underset{\displaystyle Y}{\underset{\shortparallel}{F_Y(Z)}} \text{ to be } f.$$

This is indeed a natural transformation, since for $m \in M$ we have

$m \cdot -$

$$\begin{tikzcd}
F_X(Z) \arrow[r, equal] \arrow[d, "\alpha_F"'] \arrow[rrr, bend left, "F_X(m)"] & X \arrow[r, "m\cdot-"] \arrow[d, "f"] & X \arrow[r, equal] \arrow[d, "f"'] & F_X(Z) \arrow[d, "\alpha_F"] \\
F_Y(Z) \arrow[r, equal] \arrow[rrr, bend right, "F_Y(m)"'] & Y \arrow[r, "m\cdot-"] & Y \arrow[r, equal] & F_Y(Z)
\end{tikzcd}$$

since all inner squares commute the outer one commutes.

In total we get a functor

$$\Phi : M\text{-Set} \longrightarrow \mathrm{Fun}(BM, \mathrm{Sets})$$
$$(X, \cdot) \longmapsto F_X$$
$$(X \xrightarrow{f} Y) \longmapsto F_X \xRightarrow{\alpha_f} F_Y.$$

An explicit inverse of Φ is given by the functor

$$\Psi : \mathrm{Fun}(BM, \mathrm{Sets}) \longrightarrow M\text{-Set}$$

on objects $\quad F \longmapsto F(z)$

this is canonically a $\mathrm{Maps}(F(z), F(z))$-set and in particular M-set via $F : M \to \mathrm{Maps}(F(z), F(z))$, $M = \mathrm{Hom}_{BM}(z, z)$

on morphisms

$$\alpha : F \Rightarrow G \longmapsto \alpha_z : F(z) \longrightarrow G(z).$$

Check that these two functors are indeed strict inverses.

$\boxtimes$.

Example: Let $(M, \cdot)$ be a monoid.

Then

$$\begin{array}{ccc} \text{Set-}M & \cong \; M^{op}\text{-Set} \cong & \mathrm{Fun}(\mathbb{B}(M^{op}), \mathrm{Set}) \\ & & \| \\ & & \mathrm{Fun}((\mathbb{B}M)^{op}, \mathrm{Set}) \\ & & \|\text{ Def} \\ & & \mathrm{PSh}(\mathbb{B}M) \end{array}$$

the category of right M-sets and morphisms of right M-sets

We can put together what we saw thus far:

Next we want to consider how monoid-morphisms and universal categories of monoids interact.

Lemma: Let $(M_1,\cdot),(M_2,\cdot)$ be monoids. Let $M_1 \xrightarrow{f} M_2$ be a monoid morphism. Let X be an M_2-set. Then the assignment $m_1 \in M_1$ $x \in X$ $m_1 \cdot x := f(m_1) \cdot x$ turns X into a M_1-set. We call this M_1-set $\mathrm{res}_f(X)$ the restriction of X along f.

Proof: Exercise.

Theorem: (Special case of baby version of straightening unstraightening)

Let $(M, \cdot)$ be a monoid. There are inverse isomorphisms of categories

$$\begin{array}{c} \mathrm{Cat}/_{BM} \\ \cup\!\!\!\mid \ \text{Full!! subcategory} \end{array}$$

$$\mathrm{St}^{cocart} : \mathrm{Cocart}(BM) \underset{\longleftarrow}{\overset{\cong}{\longrightarrow}} \mathrm{Fun}(BM, \mathrm{Sets}) : \mathrm{Un}^{cocart}$$

called "straightening" and "unstraightening".

For a cocartesian fibration $p : \mathcal{U} \to BM$ the value of $\mathrm{St}^{cocart}(p) : BM \longrightarrow \mathrm{Sets}$ at $z \in BM$ is the fibre $p^{-1}(z) = \mathrm{ob}\,\mathcal{U}$.

Furthermore if $M_1 \xrightarrow{f} M_2$ is a monoid morphism the unstraightening equivalence sends the precomposition functor

$$- \circ Bf : \mathrm{Fun}(BM_2, \mathrm{Sets}) \longrightarrow \mathrm{Fun}(BM_1, \mathrm{Sets})$$

to the pullback

$$(Bf)^* : \mathrm{Cocart}(BM_2) \to \mathrm{Cocart}(BM_1).$$

⌜The last part means:

Let $(M_1, \cdot), (M_2, \cdot)$ be monoids.

Let $M_1 \xrightarrow{f} M_2$ be a monoid morphism.

Let X be an M_2-set. Let $\mathrm{res}_f X$ be the M_1-set whose underlying set is X and with action

$$m_1 \in M_1 \quad x \in X \quad m_1 \cdot x := f(m_1) \cdot x$$

Then we have a pullback diagram

$$\begin{tikzcd}
U(\mathrm{res}_f X) \arrow[r] \arrow[d, "\pi_{\mathrm{res}_f X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & UX \arrow[d, "\pi_X"] \\
BM_1 \arrow[r, "Bf"] & BM_2
\end{tikzcd}$$

⌟